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An Operator Newton Method for the Stefan Problem Based on Smoothing: A Local Perspective

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The initial/Neumann boundary-value enthalpy formulation for the two-phase Stefan problem is regularized by smoothing. Known estimates predict a convergence rate of $\varepsilon^{1/2}$, and this result is extended in this paper to include the case of a (nonzero) residual in the regularized problem. A modified Newton–Kantorovich framework is established, whereby the exact solution of the regularized problem is replaced by one Newton iteration. It is shown that a consistent theory requires measure-theoretic hypotheses on the starting guess and the Newton iterate, otherwise residual decrease is not expected. The circle closes in one spatial dimension, where it is shown that the residual decrease of Newton's method correlates precisely with the $\varepsilon^{1/2}$ convergence theory. © 1990 Academic Press, Inc.

1. INTRODUCTION

The two-phase Stefan problem is an evolution model of heat conduction with change of phase. It is one of several types of degenerate parabolic models studied intensively during the last two decades (cf. [2] for elaboration and references), and is characterized as a so-called moving boundary problem. The moving boundary in this model is physically described as the front, dividing the two phases of a substance undergoing change of phase, but is most precisely described mathematically, in the case of purely bulk phenomena at a single phase change temperature, θ_0, as the set of points (moving) in physical space Ω, with associated temperature, θ_0. In this case, the enthalpy, which is composed of internal and latent energy components, is a given physical function Q of temperature discontinuous at the phase change temperature, where it is set-valued. Notice that this model also includes a change of phase which is not entirely defined by a bulk

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phenomenon at θ_0, but may occur over a bracket of temperatures. In this case, the front is, in fact, a mushy region, and a point of the substance can be associated mathematically with a given phase only by tracing the history, from an initial state, of the latent energy. In principle, this history is equivalent to tracking the moving boundary.

There is a nonlocal change of variables, $\theta \mapsto K^{-1}u$, where K is the Kirchhoff transformation, such that the evolution up to time T is governed, in the absence of sources or sinks, by the parabolic equation

$$\frac{\partial H(u)}{\partial t} - \Delta u = 0, \tag{1}$$

over the space-time domain $D = \Omega \times (0, T)$, together with appropriate initial and boundary conditions. Here, $H = Q \circ K^{-1}$, and u measures the flux energy required to raise the temperature from θ_0 to θ. Equation (1) must necessarily be understood in a distributional sense, since H is discontinuous at $K\theta_0$. The latter degeneracy suggests smoothing H, say, by constructing appropriate H_ε converging to H away from its discontinuity, and analyzing the smoothed problems,

$$\frac{\partial H_\varepsilon(u_\varepsilon)}{\partial t} - \Delta u_\varepsilon = 0. \tag{2}$$

It was shown in [4, 2] that $H_\varepsilon(u_\varepsilon) \to H(u)$ and $u_\varepsilon \to u$, with order $\sqrt{\varepsilon}$ in appropriate norms. These results are described concisely in the next section, and we provide a proof, exhibiting the constants explicitly, and allowing (2) to be solved approximately, with the right hand side equal to a residual R_ε. This result is contained in Theorem 1.

Although the regularized problems are smooth, they are nonetheless nonlinear, and a natural question is whether linearization leads to a convergent procedure. This is a particularly important question computationally, since the associated linearized problems have positive-definite, self-adjoint formulations, and are amenable to standard algorithmic procedures in numerical computation, including iterative procedures like the conjugate gradient method. Thus, this paper is devoted to the study of the linearization of (2), specifically to a modified Newton–Kantorovich framework, described in Section 3. The standard functional calculus framework is inadequate (cf. [3] for a presentation), not because of a breakdown in uniform inverse bounding, but because the Lipschitz constant of the Fréchet derivative is of order ε^{-1} without further hypotheses; its square is a factor in residual estimation. This fact necessitates a delicate study of the residual directly, as contained in Theorem 2, in relation to the sets where the starting "guess" and the Newton iterate have small measure, and subsequent hypotheses upon these sets. This intricacy imposes a somewhat different norm structure upon the residual, to guarantee the residual decrease; closure with the $\sqrt{\varepsilon}$ theory occurs only in one spatial dimension, as a consequence (cf. Corollary 4). The latter depends upon the inverse bounding result of Lemma 3.

The upshot of these investigations is that Newton type methods for two-phase Stefan problems, as applied to their regularized versions, require extreme care in their application. This fact accentuates the importance of globally convergent methods, such as those presented in [4, 5, 1], despite their relatively reduced rate of convergence when interfacing with explicit computational procedures. Finally, the reader might have expected some commentary on global Newton methods invoking continuation (cf. [3] for elaboration); this, again, is quite delicate since the measure theoretic ideas, which account for the success of a plausible local theory, do not appear to admit a natural extension along an obvious homotopy path.

A final comment about the domain Ω is in order. Unless otherwise specified, Ω is a d-dimensional, bounded, uniformly Lipschitz domain. Only in Corollary 4 is d restricted to the value $d=1$.

2. The Model and Its Regularization

We assume that a function H is prescribed, C^1 in $\mathbf{R}^1\setminus\{0\}$ and monotone increasing, with a jump discontinuity of height A at zero and derivatives satisfying

$$0<\lambda\leqslant H'(\xi)\leqslant\mu<\infty, \qquad \xi\neq 0, \tag{3a}$$

$$|H''(\xi)|\leqslant\kappa, \qquad \xi\neq 0, \tag{3b}$$

where $H'(0+)$ and $H'(0-)$, as well as $H''(0+)$ and $H''(0-)$, are assumed to exist. Normalization is chosen so that $H(0-)=0$, and the jump condition takes the form

$$H(0+)=A>0.$$

The relation with multi-valued mappings will be drawn later. Under these assumptions there is a C^1 smoothing, H_ε, satisfying, for $0<\varepsilon\leqslant\varepsilon_0$,

$$\gamma/\varepsilon\geqslant H_\varepsilon'(\xi)\geqslant\lambda>0, \qquad \xi\in\mathbf{R}, \tag{4}$$

for some positive constants γ and ε_0, and such that

$$0\leqslant H(\xi)-H_\varepsilon(\xi)\leqslant\mu\varepsilon, \qquad \xi\notin[0,\varepsilon] \tag{5a}$$

$$|J_\varepsilon(\xi)-J(\xi)|\leqslant(1+\mu/\lambda)\varepsilon, \qquad \xi\in\mathbf{R}, \tag{5b}$$

where $J_\varepsilon = H_\varepsilon^{-1}$, $J = H^{-1}$, the latter denoting the continuous left inverse of H. Properties (4) and (5) are sufficient to establish the $0(\sqrt{\varepsilon})$ convergence result of Section 1 as we shall show shortly in Theorem 1. The details of the construction are given in [2, pp. 46–48], and amount to bridging $H'(0-)$ and $H'(\varepsilon)$, by a concave quadratic arc q_ε for which $\int_0^\varepsilon q_\varepsilon(\zeta)\,d\zeta = A$, to obtain H'_ε and thence H_ε.

Before stating the approximation result, it is necessary to define, more precisely, solutions of (1) and (2), and the underlying spaces. A considerable economy of effort is achieved if the equivalent (abstract) integral equation formulation, involving the Neumann inversion operator N_0, are employed. This is also compatible with the convergence analysis. Consider then the real Sobolev space $H^1(\Omega)$, with inner product taken on functions with L^2 distribution derivatives:

$$(v, w)_{H^1} = \int_\Omega \nabla v \cdot \nabla w + \frac{1}{|\Omega|}\, j(v)\, j(w), \tag{6a}$$

where

$$j(v) = \int_\Omega v. \tag{6b}$$

The norm defined by this inner product is equivalent to the standard one. If we designate

$$F = [H^1(\Omega)]^*,$$

then N_0 is the Riesz map associated with (6); i.e., if $l \in F$, and $\langle \cdot, \cdot \rangle$ is the duality pairing, then

$$\langle l, v\rangle = (N_0 l, v)_{H^1}, \qquad v \in H^1(\Omega). \tag{7}$$

It is easily verified that N_0 is a Neumann solver; i.e., if $f \in L^2(\Omega)$, then $w = N_0 f$ satisfies

$$-\Delta w = f - \frac{1}{|\Omega|}\, j(f), \tag{8a}$$

$$\frac{\partial w}{\partial \nu} = 0, \qquad \text{on } \partial\Omega, \tag{8b}$$

$$j(w) = j(f), \tag{8c}$$

in a weak sense. These facts are documented in [2, Sect. 1.1], together with the usage of the equivalent norm on F, given by

$$\|l\|_F = \langle l, N_0 l\rangle^{1/2}. \tag{9}$$

Note that, if ρ denotes the largest eigenvalue of the positive definite compact operator $N_0|_{L^2}$, then

$$\|f\|_F \leqslant \sqrt{\rho}\ \|f\|_{L^2(\Omega)}, \qquad f \in L^2(\Omega). \tag{10}$$

The equivalent formulation of (1), incorporating initial datum u_0 and homogeneous Neumann boundary conditions, is

$$\frac{\partial N_0 H(u)}{\partial t} + u = \frac{1}{|\Omega|}\, j(u), \qquad N_0 H(u)|_{t=0} = N_0 H(u_0). \tag{11}$$

Here H has the technical meaning of a maximal monotone operator on $L^2(\Omega)$, induced by the multi-valued extension of (3) to $\mathbf{R}$; in this case, the function at 0 has the set-value

$$H(0) = [0, A]. \tag{12}$$

The functions $H(u)$ and $H(u_0)$ are understood to be appropriate selections of this operator. Regularity conditions are specified in Theorem 1 to follow.

Rather than express the equivalent form of (2) directly, we consider an approximation of (2) in the form

$$\frac{\partial N_0 H_\varepsilon(u_\varepsilon)}{\partial t} + u_\varepsilon = \frac{1}{|\Omega|}\, j(u_\varepsilon) + N_0 R_\varepsilon, \qquad u_\varepsilon|_{t=0} = u_0. \tag{13}$$

Here, R_ε is interpreted as a residual, with $R_\varepsilon = 0$ corresponding directly to the equivalent version of (2).

The choice of initial condition in (13) requires an hypothesis on the measure of the set

$$K_\varepsilon = \{x : 0 \leqslant u_0(x) \leqslant \varepsilon\}, \tag{14a}$$

specifically,

$$|K_\varepsilon| \leqslant C\varepsilon, \qquad 0 < \varepsilon \leqslant \varepsilon_0. \tag{14b}$$

Hypothesis (14) is unnecessary for the convergence result of Theorem 1 if the choice $u_\varepsilon|_{t=0} = J_\varepsilon H(u_0)$ is made, but the latter does not represent a true smoothing of the problem, and is less compatible with the linearization. The implication of (14) is that it permits the estimate

$$\begin{aligned}
\|H_\varepsilon(u_0) - H(u_0)\|_F^2 &\leqslant \rho\ \|H_\varepsilon(u_0) - H(u_0)\|_{L^2(\Omega)}^2 \\
&\leqslant \rho \left[2 \int_{0 \leqslant u_0(x) \leqslant \varepsilon} [|H_\varepsilon(u_0)|^2 + |H(u_0)|^2] + \int_\Omega (\mu\varepsilon)^2 \right] \\
&\leqslant \rho 2(\gamma^2 + A^2)\, C\varepsilon + \rho\mu^2\varepsilon^2\, |\Omega|,
\end{aligned} \tag{15}$$

where we have used (4), (5a), (10), and (12), as well as (14).

The following theorem describes the approximation theory for (11) and (13), and recalls the existence/uniqueness results. The proof is an adaptation of results in [4], but allows for the more general situation considered here. Specific constants are also derived in the convergence estimates as presented in the following.

THEOREM 1. *Suppose* $u_0 \in H^1(\Omega)$ *and that* (14) *holds. There is a unique solution pair* $[u, v]$, $v = H(u)$, *for* (11) *such that*

$$u \in L^\infty((0, T); H^1(\Omega)) \cap H^1([0, T]; L^2(\Omega)), \tag{16a}$$

$$v \in L^2(D) \cap H^1([0, T]; F). \tag{16b}$$

For a function u_ε, *in the class* (16a) *and satisfying* (13), *with residual* R_ε, *the estimates*

$$\|v_\varepsilon - v\|_{L^\infty((0,T);F)} \leqslant K_1 \sqrt{\varepsilon} + \sqrt{\lambda/2}\, \|R_\varepsilon\|_{L^2((0,T);F)}\, e^{4\rho T/\lambda}, \tag{17a}$$

$$\|u_\varepsilon - u\|_{L^2(D)} \leqslant K_2 \sqrt{\varepsilon} + \|R_\varepsilon\|_{L^2((0,T);F)}\, e^{4\rho T/\lambda}, \tag{17b}$$

hold, where, in terms of the constant C_1 *presented explicitly in* (20a) *to follow*,

$$K_1 = C_1^{1/2} T^{1/2} \exp(4\rho T/\lambda), \tag{17c}$$

$$K_2 = \sqrt{2}\left(K_1/\sqrt{\lambda} + |\Omega|^{1/2}\, T^{1/2}\left(1 + \frac{\mu}{\lambda}\right)\varepsilon_0^{1/2}\right). \tag{17d)}$$

Proof. The existence and uniqueness results are contained in [2]. For the approximation, set $v_\varepsilon = H_\varepsilon(u_\varepsilon)$ and $v = H(u)$. Subtraction of (11) from (13), multiplication by $v_\varepsilon - v$, and integration over Ω (functional operation in the case of R_ε) yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_e - v\|_F^2 + (J_\varepsilon(v_\varepsilon) - J_\varepsilon(v), v_\varepsilon - v)_{L^2(\Omega)} \\
&\quad = \frac{1}{|\Omega|}\, j(J_\varepsilon(v_\varepsilon) - J(v))\, j(v_\varepsilon - v) \\
&\qquad + (J(v) - J_\varepsilon(v), v_\varepsilon - v)_{L^2(\Omega)} + (R_\varepsilon, v_\varepsilon - v)_F \\
&\quad \leqslant \|J_\varepsilon(v_\varepsilon) - J(v)\|_F\, \|v_\varepsilon - v\|_F \\
&\qquad + \|J_\varepsilon(v) - J(v)\|_{L^2(\Omega)}\, \|v_\varepsilon - v\|_{L^2(\Omega)} + \|R_\varepsilon\|_F\, \|v_\varepsilon - v\|_F \\
&\quad \leqslant 2\rho\eta(\|J_\varepsilon(v_\varepsilon) - J_\varepsilon(v)\|^2_{L^2(\Omega)} + \|J_\varepsilon(v) - J(v)\|^2_{L^2} + \|R_\varepsilon\|^2_F) \\
&\qquad + \frac{1}{2\eta}\|v_\varepsilon - v\|_F^2 + \frac{\beta}{2}\|v_\varepsilon - v\|^2_{L^2(\Omega)} + \frac{1}{2\beta}\|J_\varepsilon(v) - J(v)\|^2_{L^2(\Omega)}
\end{aligned}$$

for arbitrary positive constants η and β, where ρ is given by (10). The inequalities

$$\|J_\varepsilon(v_\varepsilon) - J_\varepsilon(v)\|^2_{L^2(\Omega)} \leqslant \frac{1}{\lambda}(J_\varepsilon(v_\varepsilon) - J_\varepsilon(v), v_\varepsilon - v)_{L^2(\Omega)},$$

$$\|v_\varepsilon - v\|^2_{L^2(\Omega)} \leqslant \frac{\gamma}{\varepsilon}(J_\varepsilon(v_\varepsilon) - J_\varepsilon(v), v_\varepsilon - v)_{L^2(\Omega)},$$

which follow from $J'_\varepsilon \leqslant 1/\lambda$ and $H'_\varepsilon \leqslant \gamma/\varepsilon$, respectively, dictate the choices $\eta = \lambda/8\rho$ and $\beta = \varepsilon/2\gamma$, and we obtain, upon estimating the j-functional by the duality norm,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_\varepsilon - v\|^2_F + \frac{1}{2}(J_\varepsilon(v_\varepsilon) - J_\varepsilon(v), v_\varepsilon - v)_{L^2(\Omega)} \\
&\quad\leqslant \left(\frac{\lambda}{4} + \frac{\gamma}{\varepsilon}\right)\|J_\varepsilon(v) - J(v)\|^2_{L^2(\Omega)} + \frac{4\rho}{\lambda}\|v_\varepsilon - v\|^2_F + \frac{\lambda}{4}\|R_\varepsilon\|^2_F \\
&\quad\leqslant |\Omega|\left(1 + \frac{\mu}{\lambda}\right)^2\left(\frac{\lambda\varepsilon^2}{4} + \gamma\varepsilon\right) + \frac{4\rho}{\lambda}\|v_\varepsilon - v\|^2_F + \frac{\lambda}{4}\|R_\varepsilon\|^2_F.
\end{aligned} \tag{18}$$

If (18) is integrated from $\tau = 0$ to $\tau = t$, and the Gronwall inequality is applied to the resultant, we obtain

$$\begin{aligned}
&\|v_\varepsilon - v\|^2_{L^\infty((0,T);F)} + \int_0^T (J_\varepsilon(v_\varepsilon) - J_\varepsilon(v), v_\varepsilon - v)_{L^2(\Omega)}\, d\tau \\
&\quad\leqslant \left[\varepsilon C_1 T + \frac{\lambda}{2}\|R_\varepsilon\|^2_{L^2((0,T);F)}\right] e^{C_2 T},
\end{aligned} \tag{19}$$

where

$$C_1 = 2\,|\Omega|\left(1 + \frac{\mu}{\lambda}\right)^2\left(\frac{\lambda\varepsilon_0}{4} + \gamma\right) + 2\rho(\gamma^2 + A^2)C + \rho\mu^2\varepsilon_0\,|\Omega|, \tag{20a}$$

and

$$C_2 = \frac{8\rho}{\lambda}. \tag{20b}$$

The proof is completed by use of the estimate

$$\|J_\varepsilon(v_\varepsilon) - J(v)\|^2_{L^2(D)} \leq 2\,\|J_\varepsilon(v_\varepsilon) - J_\varepsilon(v)\|^2_{L^2(D)} + 2\,\|J_\varepsilon(v) - J(v)\|^2_{L^2(D)}$$
$$\leq \left\{\left(\frac{2}{\lambda}\right)\varepsilon C_1 T + \|R_\varepsilon\|^2_{L^2((0,T);F)}\right\} e^{C_2 T}$$
$$+ 2\,|\Omega|\, T\left(1 + \frac{\mu}{\lambda}\right)^2 \varepsilon^2. \quad \blacksquare$$

Remark 1. If $u_0 \in L^\infty(\Omega)$, then the solution pair $[u, v]$ of Theorem 1 may be shown to lie in $L^\infty(D) \times L^\infty(D)$, thus restricting the classes of (16a), (16b).

3. A Modified Newton–Kantorovich Framework

We shall develop such a framework for the regularized problem (2) in terms of its weak formulation. For fixed ε, F_ε is a map defined so that its unique root is a solution of the initial/boundary-value problem for (2). Specifically, the map

$$F_\varepsilon : X \to Z, \tag{21a}$$

$$X = H^1([0, T]; L^2(\Omega)) \cap L^\infty((0, T); H^1(\Omega)), \tag{21b}$$

where the norm taken on X is the maximum of the norms of the two spaces, i.e.,

$$\|u\|_X = \max\left[\left(\int_\Omega |u(\cdot, 0)|^2 + \int_D |u_t|^2\right)^{1/2}, \operatorname{ess}\sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^1(\Omega)}\right],$$

and where

$$Z = L^2((0, T); G) \times H^1(\Omega),\ G = [H^1(\Omega) \cap L^\infty(\Omega)]^*, \tag{21c}$$

is defined component-wise by $u \mapsto F_\varepsilon(u) = ([F_\varepsilon(u)]_1, [F_\varepsilon(u)]_2)$, where

$$[F_\varepsilon(u)]_1\,(\varphi) = \int_\Omega \left(\frac{\partial H_\varepsilon(u)}{\partial t}\,\varphi + \nabla u \cdot \nabla\varphi\right), \qquad \varphi \in H^1(\Omega) \cap L^\infty(\Omega), \tag{22a}$$

$$[F_\varepsilon(u)]_2 = u|_{t=0} - u_0. \tag{22b}$$

Note that $u \in X$ is weakly continuous into $H^1(\Omega)$ (cf. [2, p. 240]) so that (22b) is meaningful.

It is straightforward to see that F_ε is Lipschitz continuously Frechét differentiable on X, with $F'_\varepsilon: X \to Z$, where, for $\psi \in X$,

$$[F'_\varepsilon(u)(\psi)]_1(\varphi) = \int_\Omega \left(H'_\varepsilon(u) \frac{\partial \psi}{\partial t} \varphi + \nabla\psi \cdot \nabla\varphi \right),$$

$$\varphi \in H^1(\Omega) \cap L^\infty(\Omega), \tag{23a}$$

$$[F'_\varepsilon(u)(\psi)]_2 = \psi|_{t=0}. \tag{23b}$$

As mentioned in the Introduction, a purely functional calculus approach making use of inequalities such as

$$\|[F'_\varepsilon(u)]^{-1}\| \leqslant M_1, \qquad u \in X \tag{24a}$$

$$\|F'_\varepsilon(u) - F'_\varepsilon(v)\| \leqslant M_2 \|u - v\|, \qquad u, v \in X, \tag{24b}$$

is not capable of providing a satisfactory Newton approximation theory, since M_2 varies as ε^{-1}; M_1 is independent of ε, however, when a relaxed norm structure is chosen, as we demonstrate in Lemma 3 of the next section. Since the residual of the first Newton iterate "decreases" in norm as the square of $M_2 \|F_\varepsilon(u_\varepsilon^0)\|$ (see [3] for elaboration), it follows that any such direct functional calculus theory is inconsistent with the result of Theorem 1 in its conclusions. We shall now introduce the Newton approximations, followed by the refined hypotheses sufficient for an adequate theory.

The Newton approximations are defined in the usual way by

$$u_\varepsilon^m - u_\varepsilon^{m-1} = -[F'_\varepsilon(u_\varepsilon^{m-1})]^{-1} F_\varepsilon(u_\varepsilon^{m-1}), \qquad m \geqslant 1. \tag{25}$$

However, in this paper we are presenting an algorithm in which only one Newton approximation is computed. To describe the algorithm, let u_ε^0 be a given function in X, and let u_ε^1 solve the linear evolution equation, with weak formulation prescribed by

$$\int_\Omega \left(H'_\varepsilon(u_\varepsilon^0) \frac{\partial u_\varepsilon^1}{\partial t} \varphi + \nabla u_\varepsilon^1 \cdot \nabla\varphi \right) = 0, \qquad \varphi \in H^1(\Omega) \cap L^\infty(\Omega), \quad 0 < t \leqslant T, \tag{26a}$$

$$u_\varepsilon^1|_{t=0} = u_0. \tag{26b}$$

Then u_ε^1 is the Newton approximation, as can be seen by comparing (22), (23), and (25), the latter with $m = 1$, and setting $\psi = u_\varepsilon^1 - \psi_\varepsilon^0$ in (23).

The hypotheses employed in this section are as follows. Set

$$D_\varepsilon = \{z \in D : 0 \leqslant u_\varepsilon^0(z) \leqslant \varepsilon\}, \qquad D_\varepsilon^1 = \{z \in D : 0 \leqslant u_\varepsilon^1(z) \leqslant \varepsilon\}, \tag{27a}$$

and

$$D_\varepsilon^- = \{z \in D : u_\varepsilon^0(z)\, u_\varepsilon^1(z) < 0\}. \tag{27b}$$

Then it is assumed that

$$\max(|D_\varepsilon|, |D_\varepsilon^1|, |D_\varepsilon^-|) \leqslant C_0 \varepsilon^{5/2}, \qquad 0 < \varepsilon \leqslant \varepsilon_0. \tag{27c}$$

Remark 2. The hypotheses on D_ε and D_ε^1 essentially require concave power growth, of order 2/5, in planes normal to the free boundary; this is quite stringent, and should be compared to the linear growth condition of (14). The hypothesis on D_ε^- requires the Newton approximation u_ε^1 to adhere fairly closely to the sign properties of u_ε^0. The manner in which these hypotheses are employed will be clear from the proof of the following theorem.

THEOREM 2. *Suppose that $u_\varepsilon^0 \in X$, and that $u_\varepsilon^1 \in X$ is computed according to* (26). *If u_ε^0 and u_ε^1 satisfy* (27), *then the following residual estimate holds*:

$$\begin{aligned} \|F_\varepsilon(u_\varepsilon^1)\|_{L^2((0,T);G)} \leqslant{}& (4\gamma \sqrt{C_0}\, \varepsilon^{1/4} + 2\kappa\, \|u_\varepsilon^1 - u_\varepsilon^0\|_{L^2(D)}) \\ &\cdot \|u_\varepsilon^1 - u_\varepsilon^0\|_{H^1([0,T];L^2(\Omega))}. \end{aligned} \tag{28}$$

Here, the constants are rendered by (3b), (4), and (27c).

Proof. Inequality (28) follows from a residual representation and from (23). Specifically, we begin with

$$\begin{aligned} F_\varepsilon(u_\varepsilon^1) &= F_\varepsilon(u_\varepsilon^1) - F_\varepsilon(u_\varepsilon^0) - F_\varepsilon'(u_\varepsilon^0)(u_\varepsilon^1 - u_\varepsilon^0) \\ &= \int_0^1 [F_\varepsilon'(u_\varepsilon^0 + s(u_\varepsilon^1 - u_\varepsilon^0)) - F_\varepsilon'(u_\varepsilon^0)](u_0^1 - u_\varepsilon^0)\, ds, \end{aligned} \tag{29}$$

so that $[F_\varepsilon(u_\varepsilon^1)]_1(\varphi)$ is represented, via (29) and (23a), by

$$\int_0^1 \int_\Omega ([H_\varepsilon'(u_\varepsilon^0 + s(u_\varepsilon^1 - u_\varepsilon^0)) - H_\varepsilon'(u_\varepsilon^0)] \frac{\partial}{\partial t}(u_\varepsilon^1 - u_\varepsilon^0)\, \varphi\, ds. \tag{30}$$

An appropriate estimation of this expression proceeds by domain decomposition, i.e., by splitting D. We consider the cases separately.

(i) Suppose $z \in D_\varepsilon^-$. Then the most pessimistic estimate gives

$$\begin{aligned} K_\varepsilon(z, s) :={}& |H_\varepsilon'(u_\varepsilon^0(z) + s(u_\varepsilon^1(z) - u_\varepsilon^0(z))) - H_\varepsilon'(u_\varepsilon^0(z))| \\ \leqslant{}& \frac{2\gamma}{\varepsilon} + \kappa\, |u_\varepsilon^1(z) - u_\varepsilon^0(z)|, \end{aligned} \tag{31}$$

after addition and subtraction of $H_\varepsilon'(\varepsilon)$ within absolute values; here, κ is described in (3b).

(ii) Suppose $z \in D_\varepsilon \backslash D_\varepsilon$. Then the slightly sharper estimate

$$K_\varepsilon(z, s) \leqslant \frac{2\gamma}{\varepsilon} + \kappa s \, |u_\varepsilon^1(z) - u_\varepsilon^0(z)| \tag{32}$$

holds.

(iii) Suppose $z \in D_\varepsilon^1 \backslash D_\varepsilon$. Then the same reasoning as in (ii) leads to (32).

(iv) Suppose $z \in D \backslash (D_\varepsilon \cup D_\varepsilon \cup D_\varepsilon^1)$. Then

$$K_\varepsilon(z, s) \leqslant \kappa s \, |u_\varepsilon^1(z) - u_\varepsilon^0(z)| \tag{33}$$

holds, since the segment connecting $u_\varepsilon^0(z)$ and $u_\varepsilon^1(z)$ does not intersect $(0, \varepsilon)$.

The case distinctions (i)–(iv) permit the proof to continue. If the representation (30) is squared and integrated from 0 to T, we obtain, after a twofold application of the Schwarz inequality,

$$\begin{aligned}
\int_0^T |[F_\varepsilon(u_\varepsilon^1)]_1(\varphi)|^2 \leqslant 2 \int_0^1 & \left\{ \int_{D_\varepsilon \cup D_\varepsilon^1 \cup D_\varepsilon} K_\varepsilon^2(z, s)\, dz \right. \\
& \left. + \int_{D \backslash (D_\varepsilon \cup D_\varepsilon^1 \cup D_\varepsilon)} K_\varepsilon^2(z, s)\, dz \right\} ds \\
& \times \left\| \frac{\partial}{\partial t} (u_\varepsilon^1 - u_\varepsilon^0) \right\|_{L^2(D)}^2 \|\varphi\|_{L^2(\Omega)}^2 \\
\leqslant 4 & \left(\frac{4\gamma^2}{\varepsilon^2} \int_{D_\varepsilon \cup D_\varepsilon^1 \cup D_\varepsilon} dz + \kappa^2 \int_D |u_\varepsilon^1 - u_\varepsilon^0|^2 \, dz \right) \\
& \cdot \left\| \frac{\partial}{\partial t} (u_\varepsilon^1 - u_\varepsilon^0) \right\|_{L^2(D)}^2 \|\varphi\|_{L^2(\Omega)}^2.
\end{aligned} \tag{34}$$

The proof is now concluded upon application of (27c) to (34). ∎

For technical reasons, the route now taken does not pass through inequality (24a), which, in conjunction with Theorem 2, would translate a duality estimate on $F_\varepsilon(u_\varepsilon^0)$ into a corresponding estimate on $F_\varepsilon(u_\varepsilon^1)$. Rather, a stronger estimate on $F_\varepsilon(u_\varepsilon^0)$ is required, viz., an L^2 estimate, so that the time derivative and the Laplacian perturbation can be adequately inverted. This is presented in Lemma 3 of the next section.

4. The Fundamental Lemma and Corollary

The purpose of this section is to derive the following lemma and its consequence in terms of residual estimation.

Lemma 3. *Let $u \in X$ (cf. (21b)). Then, for each $[g, w] \in L^2(D) \times H^1(\Omega)$ the system*

$$[F_\varepsilon'(u)\psi]_1 = g, \qquad \psi|_{t=0} = w, \tag{35}$$

is uniquely solvable for $\psi \in X$, and, in terms of the standard norm on $H^1(\Omega)$,

$$\|\psi\|_X \leqslant \sqrt{2} \max\left(\frac{2}{\lambda}, 1\right) \exp(T/2) \, \|[g, w]\|_{L^2(D) \times H^1(\Omega)}. \tag{36}$$

It follows that, if $F_\varepsilon(u_\varepsilon^0) \in L^2(D) \times H^1(\Omega)$, then

$$\|F_\varepsilon(u_\varepsilon^1)\|_{L^2((0,T);G)} \leqslant c(\varepsilon^{1/4} + \|F_\varepsilon(u_\varepsilon^0)\|_{L^2(D) \times H^1(\Omega)}) \, \|F_\varepsilon(u_\varepsilon^0)\|_{L^2(D) \times H^1(\Omega)}, \tag{37a}$$

where

$$c = c_0(4\gamma \sqrt{C_0} + 2\kappa c_0) \tag{37b}$$

and C_0 is given by (27c), with c_0 given by

$$c_0 = \sqrt{2} \max\left(\frac{2}{\lambda}, 1\right) \exp(T/2). \tag{37c}$$

Proof. We use the method of horizontal lines, as employed in [2], to discretize (35), with the left hand side defined by (23), except that $\varphi \in H^1(\Omega)$. Specifically, let an arbitrary uniform partition, $0 = t_0 < t_1 < \cdots < t_M = T$, of $[0, T]$ be specified, with $t_k - t_{k-1} = \Delta t$, $k = 1, ..., M$.

We consider the sequence of problems

$$\omega_{k,\varepsilon}(\psi_k - \psi_{k-1}) \, \Delta t^{-1} - \Delta \psi_k = \bar{g}_k, \qquad \psi_0 = w, \tag{38}$$

where (38) is understood as holding in the distributional sense, i.e., in F, and where

$$\omega_{k,\varepsilon} = H_\varepsilon'(u(\cdot, t_k)) \geqslant \lambda > 0, \tag{39a}$$

$$\bar{g}_k = \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} g(\cdot, t) \, dt. \tag{39b}$$

The existence of a unique solution $\psi_k \in H^1(\Omega)$ of (38) follows from minimizing the quadratic functional,

$$G(f) = \frac{1}{2}\int_\Omega |\nabla f|^2 + \frac{1}{2\,\Delta t}\int_\Omega \omega_{k,\varepsilon}\,|f|^2 - \frac{1}{\Delta t}\int_\Omega \omega_{k,\varepsilon}\psi_{k-1} f - \int_\Omega \bar{g}_k f, \tag{40}$$

over $H^1(\Omega)$. By selecting the test function,

$$f = \psi_k - \psi_{k-1},$$

for the distribution in (38) we find, upon summing on $k = 1, ..., m$,

$$\begin{aligned}
&\lambda \sum_{k=1}^{m} \left\| \frac{\psi_k - \psi_{k-1}}{\Delta t} \right\|^2_{L^2(\Omega)} \Delta t + \frac{1}{2}\,\|\nabla\psi_m\|^2_{L^2(\Omega)} \\
&\quad \leqslant \|\nabla w\|^2_{L^2(\Omega)} + \sum_{k=1}^{m} \|\bar{g}_k\|_{L^2(\Omega)}\,\|\psi_k - \psi_{k-1}\|_{L^2(\Omega)} \\
&\quad \leqslant \|\nabla w\|^2_{L^2(\Omega)} + \frac{1}{2\lambda}\sum_{k=1}^{m} \|\bar{g}_k\|^2_{L^2(\Omega)}\,\Delta t + \frac{\lambda}{2}\sum_{k=1}^{m} \|\psi_k - \psi_{k-1}\|^2_{L^2(\Omega)}/\Delta t
\end{aligned} \tag{41}$$

so that (41) leads to

$$\frac{\lambda}{2}\sum_{k=1}^{m} \left\| \frac{\psi_k - \psi_{k-1}}{\Delta t} \right\|^2_{L^2(\Omega)} \Delta t + \frac{1}{2}\,\|\nabla\psi_m\|^2_{L^2(\Omega)} \leqslant \|\nabla w\|^2_{L^2(\Omega)} + \frac{1}{2\lambda}\,\|g\|^2_{L^2(D)}. \tag{42}$$

To obtain an estimate for $\sup_{1 \leqslant m \leqslant M} \|\psi_m\|^2_{L^2(\Omega)}$, select the test function $f = \psi_k$ in (38). Similar arguments lead to

$$\begin{aligned}
&\left(\frac{\lambda}{2}\right) \|\psi_m\|^2_{L^2(\Omega)} + \sum_{k=1}^{m} \|\nabla\psi_k\|^2_{L^2(\Omega)}\,\Delta t \\
&\qquad \leqslant \left(\frac{\lambda}{2}\right) \|w\|^2_{L^2(\Omega)} + \frac{1}{\lambda}\,\|g\|^2_{L^2(D)} + \frac{\lambda}{4}\sum_{k=1}^{m} \|\psi_k\|^2_{L^2(\Omega)}\,\Delta t.
\end{aligned} \tag{43}$$

Estimates (42) and (43) and the discrete Gronwall inequality [2, pp. 52–53], together with the definitions of norms, imply that the piecewise linear function $\Psi_{\Delta t}$, defined by the interpolation

$$\Psi_{\Delta t}(t_k) = \psi_k, \tag{44}$$

satisfies

$$\|\Psi_{\Delta t}\|^2_X \leqslant \max\left(\frac{4}{\lambda}, 2\right)\left(\|w\|^2_{H^1(\Omega)} + \frac{1}{\lambda}\,\|g\|^2_{L^2(D)}\right)\exp(T). \tag{45}$$

Now it is shown in [2, Chap. 5] that appropriate cluster points of the sequence (44) must be solutions of (35); the situation is even more elementary here since (35) is linear. Uniqueness is elementary, since the difference $\tilde{\psi}$ of any two solutions solves a smooth problem, and hence $(\nabla\tilde{\psi})_t \in L^2(D)$. This allows the relation, for $D_t = (0, t) \times \Omega$, $t \leqslant T$,

$$\int_{D_t} |H_\varepsilon'(u)\,\tilde{\psi}_t|^2 + \frac{1}{2}\int_{D_t} \frac{d}{dt}\,|\nabla\tilde{\psi}|^2 = 0,$$

hence $\tilde{\psi} \equiv 0$. Invertibility is now demonstrated, and (36) follows directly from (45). Inequality (37) is a direct consequence of Theorem 2 and the X-norm definition. ∎

A direct corollary of Theorem 1 and Lemma 3 is the following.

COROLLARY 4. *Suppose that the initial residual, $F_\varepsilon(u_\varepsilon^0)$, satisfies*

$$\|F_\varepsilon(u_\varepsilon^0)\|_{L^2(D)\times H^1(\Omega)} \leqslant c_1\,\varepsilon^{1/4}, \tag{46}$$

for c_1 independent of ε, and that Ω is a one-dimensional interval. Then, in the notation of Theorem 1, *with $v_\varepsilon = v_\varepsilon^1$ and $u_\varepsilon = u_\varepsilon^1$,*

$$\|v_\varepsilon - v\|_{L^\infty((0,T);F)} \leqslant C_1\sqrt{\varepsilon}, \tag{47a}$$

$$\|u_\varepsilon - u\|_{L^2(D)} \leqslant C_1\sqrt{\varepsilon}, \tag{47b}$$

where C_1 is independent of ε.

Proof. Identify R_ε of Theorem 1 with $F_\varepsilon(u_\varepsilon^1)$. The assumption that Ω is one-dimensional is used precisely to identify the dual space F, of $H^1(\Omega)$, with the dual space G, of $H^1(\Omega) \cap L^\infty(\Omega)$, with equivalent norms. In this case, inequality (37) of Lemma 3 may be used in conjunction with hypothesis (46), to estimate R_ε in (17a), (17b), thence leading to (47a), (47b). ∎

5. CLOSING REMARKS

The author thanks the referee for citing the related reference [6], and for suggesting the inclusion of certain effects, not mentioned earlier in the paper, which we shall comment upon now. Resonance and bifurcation were the possible effects suggested; the first of these can be handled fairly easily, whereas the second raises extremely interesting issues. The addition of a time dependent term to the rhs of (1) does not change the error estimate of Theorem 1, and the possible growth in time is already reflected in the estimates (37), which appear implicitly in (47). Regarding bifurcation, though this model is logically separate, since it is not uniqueness which breaks down as $\varepsilon \to 0$, but rather differentiability, one can still discern clear parallels. One can think of the map $F(\varepsilon, u) = F_\varepsilon(u)$ in terms of its parametric dependence upon ε; if one has already identified a solution branch, then it is conceivable that one can track such a branch, even in the face of operator singularities, so long as the approximations are sufficiently delicate in terms of measure theoretic properties. This is the gist of Theorem 2.

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